

Decomposition of Positive Sesquilinear Forms and the Central Decomposition of Gauge-Invariant Quasi-Free States on the Weyl-Algebra

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Z. Naturforsch. **45a**, 17–28 (1990); received June 10, 1989

A decomposition theory for positive sesquilinear forms densely defined in Hilbert spaces is developed. On decomposing such a form into its closable and singular part and using Bochner's theorem it is possible to derive the central decomposition of the associated gauge-invariant quasi-free state on the boson C^* -Weyl algebra. The appearance of a classical field part of the boson system is studied in detail in the GNS-representation and shown to correspond to the so-called singular subspace of a natural enlargement of the one-boson testfunction space. In the example of Bose-Einstein condensation a non-trivial central decomposition (or equivalently a non-trivial classical field part) is directly related to the occurrence of the condensation phenomenon.

Key words: Closable and singular positive sesquilinear forms; gauge-invariant quasi-free states on the Weyl algebra; central decomposition; classical part of boson fields; condensation phenomena.

1. Introduction

From the very idea of Bose-Einstein condensation, that the zero momentum is macroscopically occupied, it follows that the thermodynamic equilibrium distribution should have some kind of singular behaviour. This was in fact confirmed by rigorous investigations of various forms of the Bose-Einstein condensation, where the singularity only appears in the infinite volume limit with fixed particle density, which should be larger than some critical value. In [1], [2] and [3] (see also the fundamental work in this field [4]), in the formalism of operator algebraic quantum statistical mechanics for this kind of thermodynamic limit the limiting Gibbs states have been deduced, each of which is gauge-invariant and quasi-free and hence uniquely determined by a positive sesquilinear form defined on the one-boson testfunction space. Below the critical density these sesquilinear forms are closable, whereas above the critical density there is an additional singular sesquilinear form, which is given by the evaluation of the testfunctions at zero momentum and thus yields the non-closability of the total positive form. Hence the occurrence of condensation is closely related to the occurrence of a non-closable positive

sesquilinear form in the limiting Gibbs state of the infinite Bose system.

In the present work we generalize the above situation. First we investigate the decomposition of an arbitrary positive sesquilinear form t (defined on a pre-Hilbert space E) into its closable and singular part. Then we use this decomposition of t to deduce the central decomposition of the gauge-invariant quasi-free state ω on the Weyl algebra over E associated with the form t . There is a one-to-one correspondence between the appearance of a non-trivial singular part of t and a non-trivial central decomposition of ω . Thus condensation appears if and only if the central decomposition becomes non-trivial, and the latter is only possible for infinite boson systems. In the usual statistical interpretation, the central measure gives the classical-statistical mixture of the ensemble into disjoint primary (purely quantum mechanical) states. The disintegration via the central measure (spatial decomposition) of the weak closure of the GNS-represented Weyl algebra appears in the form of a tensor product, one factor of which being a factorial von Neumann algebra and the other one being merely commutative. Hence in the reconstructed quantum mechanics (GNS-representation of ω) the mentioned classical-statistical mixture – which also represents the collective phenomenon of condensation – is expressed by an additional classical field. The latter arises from the singular part of t , whereas the purely quantum

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mechanical one (the factorial algebra) is related to the closable part of t . In the above case of Bose-Einstein condensation the classical phase space is two dimensional and can be parametrized in terms of modified polar coordinates, which are interpreted as the (scaled) particle density of the condensate, R , and a phase angle, ϑ , which is in connection with the gauge transformations. Due to the statistical interpretation, each (infinite) boson system of the ensemble has sharp values R and ϑ and is described by the state $\omega_{R, \vartheta}$ appearing in the central decomposition of ω : $\omega = \int_{R=0}^{\infty} \int_{\vartheta=0}^{2\pi} \omega_{R, \vartheta} d\mu(R, \vartheta)$.

The classical features of the boson field also can be seen on the C^* -algebraic level. The decomposition theory of positive sesquilinear forms (cf. Sect. 2) allows an enlargement of the one-boson testfunction space E , to a Hilbert space \mathcal{H} and consequently suggests also a natural enlargement of the C^* -Weyl algebra (which now is defined by means of a degenerate symplectic form on \mathcal{H}), so that the classical field part may vary independently of the quantum mechanical one. The enlargement of the algebra is adapted to ω , and thus ω may be extended to this new C^* -Weyl algebra. Due to the orthogonal decomposition of \mathcal{H} into its closable and singular subspaces (see Sect. 2) the new Weyl algebra divides into a tensor product, and so does the extended ω . In this way the C^* -boson system divides tensorially into a quantum mechanical C^* -system and a classical (commutative) one, whose GNS-represented weak closures agree with the above mentioned factorial and commutative von Neumann algebras. Now the field operators associated with the GNS-representation of the enlarged Weyl algebra are well defined, and the merely quantum mechanical and the merely classical field expressions are approximable by letting go the testfunctions in the argument of the (smeared) field operators from E into the closable or singular subspace of \mathcal{H} , respectively.

A similar procedure is also possible for the class of coherent states. We refer to [5], where the macroscopic (classical) aspects of coherent light with a high photon density has been discussed.

As pointed out above, the concept of the decomposition of positive sesquilinear forms with the closable and singular subspaces is essential for the study of both the purely quantum mechanical and the classical parts of boson fields within the classes of the gauge-invariant quasi-free states and the coherent states. One may ask, whether this concept also applies to other classes of states. Indeed, in [6] methods basing on the

developments of Sect. 2 are indicated to treat the classical states with finitely many (eventually unbounded) modes. The general class of the classical states is defined in terms of an operator algebraic generalization of Gauber's P -representation with positive measures and includes the coherent and gauge-invariant quasi-free states. For the classical states the really macroscopic (classical) modes may be identified by searching for the singular subspace. In the frame of classical states it seems to be possible to study in a rigorous way some classical non-linear dynamical equations of quantum optics, which now are basing on a complete microscopic theory and not on the usual heuristic arguments. These dynamical equations one may get from microscopic quantum interactions by the above procedure, a suitable enlargement of the one-boson testfunction space and a subsequent separation of the global classical dynamics by letting go the testfunctions into the singular subspace. The indicated approach to this field of quantum optics would have the advantage of studying both the quantum mechanical and the classical aspects of such dynamics and their dynamical interaction. (For references see the citations in [6].)

After this outlook we return to the present investigations and proceed as follows. Section 2 is devoted to a detailed analysis of the above mentioned decomposition of a positive sesquilinear form t defined on a pre-Hilbert space E . In contrast to closability of a positive form (which then arises from a positive self-adjoint operator), we introduce the notion for a form to be singular. The basic idea of the decomposition of t , as given in [7], is the completion of E with respect to a stronger hilbertian norm, namely the sum of the old one and the quadratic form, followed by an orthogonal decomposition of this completion into the so-called closable and singular subspaces. In the standard literature on quadratic forms such completions are usually only considered for closable forms. In the context of real symplectic spaces similar completions and decompositions are done in [8]; they agree with those here if the imaginary part of the inner product of E is identified with the symplectic form. But because E is an inner product space (and hence closability and singularity of forms are naturally defined in terms of the inner product topology), the structure of the decomposition theory becomes here much richer than in the symplectic case. Also a comparison of positive forms is introduced, which among other results shows that the above decomposition of t is the one with the largest closable part and the smallest possible singular

one, and is unique in this sense. Finally, for applications of the theory some examples of decompositions are given.

After some preliminary results on states on the Weyl algebra $\mathcal{W}(E)$ over E we treat in Sect. 3 the central decomposition of gauge-invariant quasi-free states. Using the decomposition of t from Sect. 2, the GNS-representation of the gauge-invariant quasi-free state ω associated with t is easily constructed, where some arguments go even back to [4]. Employing Bochner's theorem for the exponential of the singular part of t ensures the existence of a measure on the characters of E , which lifts to a regular Borel measure on the state space of $\mathcal{W}(E)$ decomposing ω . The latter is shown to be the central measure of ω (cf. also [8]). Doing this requires more than showing the mutual disjointness of the supporting states [9] and is achieved by an explicit calculation of the Tomita map. In the case of a finite dimensional singular part of t by means of Fourier transformation the central measure is constructed explicitly and applied to the above mentioned limiting Gibbs state of the Bose-Einstein condensation.

2. Decomposition of Positive Sesquilinear Forms

Let \mathcal{H} be a fixed Hilbert space over \mathbb{K} with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$, where \mathbb{K} denotes the real field \mathbb{R} or the complex field \mathbb{C} . Let t be a symmetric sesquilinear form in \mathcal{H} , which is defined on the dense domain $E := \mathcal{D}(t)$. Usually t is called closable, if $\lim_{n \rightarrow \infty} \|f_n\| = 0$ and $\lim_{n, m \rightarrow \infty} t(f_n - f_m, f_n - f_m) = 0$ implies $\lim_{n \rightarrow \infty} t(f_n, f_n) = 0$. The importance of closable sesquilinear forms depends on the fact that (with some additional conditions) they arise from operators ([10], Chapter VI). In contrast to the notion of closability we define:

Definition 2.1. *The symmetric sesquilinear form $t: E \times E \rightarrow \mathbb{K}$ is called singular (with respect to the scalar product $\langle \cdot, \cdot \rangle$ on E), if for each $f \in E$ there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in E , such that $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$ and $\lim_{n \rightarrow \infty} t(f_n, f_n) = 0$.*

From now on we assume t to be positive (that is: $t(f, f) \geq 0 \ \forall f \in E$) and use the term *positive form* instead of *positive symmetric sesquilinear form*. On E we

define a new scalar product by

$$E \times E \rightarrow \mathbb{K}, \quad (f, g) \mapsto \langle f, g \rangle + t(f, g). \quad (2.1)$$

The completion of E with respect to this scalar product will be denoted by $(\mathcal{H}, (\cdot | \cdot)_{\mathcal{H}})$. That is: there is a linear operator $U: E \rightarrow \mathcal{H}$, such that $U(E)$ is dense in \mathcal{H} and $(Uf | Ug)_{\mathcal{H}} = \langle f, g \rangle + t(f, g) \ \forall f, g \in E$. A linear subspace E_0 of E is called a form core for t , if $U(E_0)$ is dense in \mathcal{H} . If t' is another positive form in \mathcal{H} with domain E' , we say t' is dominated by t (and write $t' \leq t$), if $E \subseteq E'$, E is a form core for t' and $t'(f, f) \leq t(f, f) \ \forall f \in E$.

Since by (2.1) one has $\|Uf\|_{\mathcal{H}} \geq \|f\| \ \forall f \in E$, the map $U^{-1}: U(E) \rightarrow E$, $U(f) \mapsto f$ has a contractive extension $V: \mathcal{H} \rightarrow \mathcal{H}$. Then $\mathcal{H}_s := \{\eta \in \mathcal{H} \mid V\eta = 0\} = \ker(V)$ is a closed subspace of \mathcal{H} , called the *singular subspace* (corresponding to t).

Lemma 2.2. *For the positive form t the following conditions are equivalent:*

- (i) t is closable;
- (ii) if $(f_n)_{n \in \mathbb{N}}$ is any sequence in E so that $\lim_{n \rightarrow \infty} \|f_n\| = 0$ and $(Uf_n)_{n \in \mathbb{N}}$ is Cauchy in \mathcal{H} , then $\lim_{n \rightarrow \infty} \|Uf_n\|_{\mathcal{H}} = 0$;
- (iii) $\mathcal{H}_s = \{0\}$, i.e. V is injective.

Proof: Transferring by U the notion of closability to \mathcal{H} gives (i) \Leftrightarrow (ii).

(ii) \Rightarrow (iii): Let $\eta \in \mathcal{H}_s$ and $(f_n)_{n \in \mathbb{N}}$ a sequence in E with $\lim_{n \rightarrow \infty} \|Uf_n - \eta\|_{\mathcal{H}} = 0$. From the continuity of V one gets $0 = V\eta = \lim_{n \rightarrow \infty} VUf_n = \lim_{n \rightarrow \infty} f_n$ and consequently by assumption $\eta = 0$.

(iii) \Rightarrow (ii): Let $(f_n)_{n \in \mathbb{N}}$ be any sequence in E so that $\lim_{n \rightarrow \infty} \|f_n\| = 0$ and $(Uf_n)_{n \in \mathbb{N}}$ is Cauchy in \mathcal{H} with limit $\xi \in \mathcal{H}$. Then $V\xi = \lim_{n \rightarrow \infty} VUf_n = \lim_{n \rightarrow \infty} f_n = 0$, from which $\xi \in \ker(V) = \mathcal{H}_s = \{0\}$ follows. \square

Obviously, if t is not closable, \mathcal{H}_s contains just those vectors of \mathcal{H} , which prevent t from being closable. Put $\mathcal{H}_c := \mathcal{H}_s^\perp$ (\mathcal{H}_s^\perp is the orthogonal complement of \mathcal{H}_s in \mathcal{H}) – the *closable subspace* – and denote the orthogonal projections from \mathcal{H} onto \mathcal{H}_c and \mathcal{H}_s by P_c and P_s , respectively. Of course we have $VP_c = V$. Because of $\|f\| = \|VUf\| = \|VP_c Uf\| \leq \|P_c Uf\|_{\mathcal{H}} \ \forall f \in E$ we can define two positive forms, t_c and t_s , with domain E by

$$t_c(f, g) := (Uf | P_c U g)_{\mathcal{H}} - \langle f, g \rangle \quad (2.2)$$

and

$$t_s(f, g) := (Uf | P_s U g)_{\mathcal{H}}$$

for all $f, g \in E$. Obviously $t = t_c + t_s$.

Lemma 2.3. *For the positive form t the following conditions are equivalent:*

- (i) *If t' is a closable positive form with $t' \leq t$, then $t' \equiv 0$;*
- (ii) $t = t_s$;
- (iii) *the restriction of V to \mathcal{K}_c is a unitary onto \mathcal{H} ;*
- (iv) t is singular;
- (v) *for each $f \in \mathcal{H}$ there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in E with $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$ and $\lim_{n \rightarrow \infty} t(f_n, f_n) = 0$.*

Proof: (i) \Rightarrow (ii): If $t = t_c + t_s$, then $t_c \leq t$. Consequently $t_c \equiv 0$.

(ii) \Rightarrow (iii): If η is any vector in \mathcal{K}_c and $\lim_{n \rightarrow \infty} \|Uf_n - \eta\|_{\mathcal{H}} = 0$, $f_n \in E$, then $V\eta = \lim_{n \rightarrow \infty} VUf_n = \lim_{n \rightarrow \infty} f_n$. Moreover, by assumption and (2.2)

$$\begin{aligned} \| \eta \|_{\mathcal{H}}^2 &= \lim_{n \rightarrow \infty} \| Uf_n \|_{\mathcal{H}}^2 = \lim_{n \rightarrow \infty} (\| P_c Uf_n \|_{\mathcal{H}}^2 + \| P_s Uf_n \|_{\mathcal{H}}^2) \\ &= \lim_{n \rightarrow \infty} \| f_n \|^2 = \| V\eta \|^2. \end{aligned}$$

(iii) \Rightarrow (iv): Let $f \in E$, then there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in E with $\lim_{n \rightarrow \infty} Uf_n = P_c Uf$. Hence $\lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} VUf_n = VP_c Uf = VUf = f$. Consequently by assumption $\lim_{n \rightarrow \infty} t(f_n, f_n) = \lim_{n \rightarrow \infty} (\| Uf_n \|_{\mathcal{H}}^2 - \| f_n \|^2) = \| P_c Uf \|^2 - \| f \|^2 = 0$.

(iv) \Rightarrow (i): Assume t' to be a closable positive form with $t' \leq t$. By (iv) for $f \in E$ let $(f_n)_{n \in \mathbb{N}}$ be a sequence with $\lim_{n \rightarrow \infty} f_n = f$ and $\lim_{n \rightarrow \infty} t(f_n, f_n) = 0$. Hence $\lim_{n, m \rightarrow \infty} t(f_n - f_m, f_n - f_m) = 0$ and because of $t' \leq t$ one has $\lim_{n, m \rightarrow \infty} t'(f_n - f_m, f_n - f_m) = 0$. Since t' is closable we compute $0 \leq t'(f, f) = \lim_{n \rightarrow \infty} t'(f_n, f_n) \leq \lim_{n \rightarrow \infty} t(f_n, f_n) = 0$.

(iv) \Leftrightarrow (v) is obvious. \square

For the decomposition $t = t_c + t_s$ of (2.2) we obtain the essential property:

Theorem 2.4. *The form t_c is closable and t_s is singular.*

Proof: If $(f_n)_{n \in \mathbb{N}}$ is any sequence in E with $\lim_{n \rightarrow \infty} \|f_n\| = 0$ and $\lim_{n, m \rightarrow \infty} t_c(f_n - f_m, f_n - f_m) = 0$, then $(P_c Uf_n)_{n \in \mathbb{N}}$ is Cauchy in \mathcal{H} converging to some vector $\eta \in \mathcal{K}_c$. Consequently $V\eta = \lim_{n \rightarrow \infty} VP_c Uf_n = \lim_{n \rightarrow \infty} VUf_n = \lim_{n \rightarrow \infty} f_n = 0$. Hence $\eta = 0$ since V is injective on \mathcal{K}_c . Thus $\lim_{n \rightarrow \infty} t_c(f_n, f_n) = \lim_{n \rightarrow \infty} (\| P_c Uf_n \|_{\mathcal{H}}^2 - \| f_n \|^2) = 0$ and t_c is closable.

Let $f \in E$ and $(f_n)_{n \in \mathbb{N}}$ be a sequence in E such that $\lim_{n \rightarrow \infty} Uf_n = P_c Uf$. Then $\lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} VUf_n = VP_c Uf = VUf = f$ and $\lim_{n \rightarrow \infty} t_s(f_n, f_n) = \lim_{n \rightarrow \infty} \| P_s Uf_n \|_{\mathcal{H}}^2 = 0$ and t_s is singular. \square

The above theorem suggests the following definition:

Definition 2.5. *The form t_c is called the closable part, and t_s is called the singular part of the positive form t .*

To the closure \bar{t}_c ([10], Theorem VI.1.17) of the closable part t_c there is uniquely associated a positive self-adjoint operator T_c in \mathcal{H} such that $\mathcal{D}(T_c^{\frac{1}{2}}) = \mathcal{D}(\bar{t}_c)$ and

$$\bar{t}_c(f, g) = \langle T_c^{\frac{1}{2}} f, T_c^{\frac{1}{2}} g \rangle \quad \forall f, g \in \mathcal{D}(\bar{t}_c),$$

and further E is a core for $T_c^{\frac{1}{2}}$ ([10], Theorem VI.2.23). We now relate T_c to the objects introduced so far.

Proposition 2.6. *V restricted to \mathcal{K}_c is a bijection onto $\mathcal{D}(\bar{t}_c)$, and $(V|_{\mathcal{K}_c})^{-1}$ is the closure $\bar{P}_c \bar{U}$ of the map $P_c U$. Moreover $T_c = \bar{P}_c \bar{U}^* \bar{P}_c \bar{U} - \mathbf{1}$.*

Proof: $V|_{\mathcal{K}_c}$ is injective. Now let $\xi \in \mathcal{K}_c$ and $(f_n)_{n \in \mathbb{N}}$ a sequence in E with $\lim_{n \rightarrow \infty} \|P_c Uf_n - \xi\|_{\mathcal{H}} = 0$ and $V\xi =: f$. Then

$$\lim_{n \rightarrow \infty} \|VP_c Uf_n - f\| = \lim_{n \rightarrow \infty} \|f_n - f\| = 0$$

and

$$\lim_{n, m \rightarrow \infty} t_c(f_n - f_m, f_n - f_m) = 0,$$

from which $f \in \mathcal{D}(\bar{t}_c)$ follows. Conversely, let $f \in \mathcal{D}(\bar{t}_c)$ and $(f_n)_{n \in \mathbb{N}}$ a sequence in E with $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$ and

$\lim_{n \rightarrow \infty} \bar{t}_c(f_n - f, f_n - f) = 0$. Then $(P_c Uf_n)_{n \in \mathbb{N}}$ is Cauchy in \mathcal{K}_c with limit $\xi \in \mathcal{K}_c$. Hence $V\xi = \lim_{n \rightarrow \infty} VP_c Uf_n = \lim_{n \rightarrow \infty} f_n = f$. The rest follows from $\langle T_c^{\frac{1}{2}} f, T_c^{\frac{1}{2}} g \rangle + \langle f, g \rangle = \bar{t}_c(f, g) + \langle f, g \rangle = (\bar{P}_c \bar{U} f | \bar{P}_c \bar{U} g)_{\mathcal{H}}$ for $f, g \in \mathcal{D}(\bar{t}_c)$. \square

Thus, by $t = t_c + t_s$ we have found a decomposition of the positive form t into a closable positive form t_c and a singular positive form t_s . We remark that the decomposition of a positive form into a closable and a singular positive form is not unique, which is seen from the following simple example: Let $[a, b]$ be a finite closed interval of the real line and $\mathcal{H} := L^2([a, b])$ and

$$t_1(f, g) := \int_a^b \overline{f'} g' \, dx \quad \text{and}$$

$$t_2(f, g) := \overline{f(c)} g(c) \text{ for some } c \in [a, b],$$

where $\mathcal{D}(t_1) = \mathcal{D}(t_2)$ is the set of all $f \in \mathcal{H}$ such that f is absolutely continuous on $[a, b]$ with derivative $f' \in \mathcal{H}$.

It is well known that t_1 is closed. t_2 is singular (see Proposition 2.11 below). But t_2 is relatively bounded with respect to t_1 with t_1 -bound zero ([10], Examples IV.1.8 and VI.1.36). Consequently, by [10], Theorem VI.1.33 the form $t = t_1 + t_2$ is closed.

But in some other sense our above decomposition $t = t_c + t_s$ is unique. For this we state a result which gets proved after Proposition 2.9:

Lemma 2.7. *If t' is any closable form satisfying $t' \leq t$, then it follows $t' \leq t_c$.*

Let $t = t' + t''$ be an arbitrary decomposition of t into a closable positive form t' and a singular positive form t'' . From Lemma 2.7 it follows $t' \leq t_c$ and therefore $t_s \leq t''$. Hence the decomposition $t = t_c + t_s$ is unique in the sense that t_c is the largest (with respect to \leq) closable positive form, or equivalently t_s is the lowest singular positive form of all possible decompositions of t into a closable and a singular positive form. We remark that the existence of the largest closable positive form smaller than t is well known and e.g. proved by different techniques in [11], Theorem S.15.

Assume that t' is a positive form with domain E' satisfying $t' \leq t$. The completion of E' with respect to the analogous scalar product as (2.1) we denote by $(\mathcal{H}', (\cdot|\cdot)_{\mathcal{H}'})$ with the embedding $U': E' \rightarrow \mathcal{H}'$ such that $U'(E')$ is dense in \mathcal{H}' and $(U'f|U'g)_{\mathcal{H}'} = \langle f, g \rangle + t'(f, g) \forall f, g \in E'$. The contractive extension of U'^{-1} we denote by $V': \mathcal{H}' \rightarrow \mathcal{H}$. Further $\mathcal{H}'_s := \ker(V')$, $\mathcal{H}'_c := \mathcal{H}'_s^\perp$ and P'_s, P'_c the orthogonal projections of \mathcal{H}' onto these subspaces. Since $\|U'f\|_{\mathcal{H}'} \leq \|Uf\|_{\mathcal{H}} \forall f \in E$ there is a selfadjoint operator C on \mathcal{H} with

$$0 \leq C \leq \mathbf{1}_{\mathcal{H}} \quad \text{and} \quad (Uf|U'g)_{\mathcal{H}'} = (Uf|CUg)_{\mathcal{H}} \quad \forall f, g \in E. \quad (2.3)$$

Therefore the map $U'f \mapsto C^{\frac{1}{2}}Uf, f \in E$ has a continuous extension to an isometry $I: \mathcal{H}' \rightarrow \mathcal{H}$.

Lemma 2.8. *In the above situation $C^{\frac{1}{2}}(\mathcal{H}'_s) \subseteq I(\mathcal{H}'_s)$.*

Proof: Let $\eta \in \mathcal{H}'_s$ and $(f_n)_{n \in \mathbb{N}}$ a sequence in E with $\lim_{n \rightarrow \infty} \|Uf_n - \eta\|_{\mathcal{H}'} = 0$. Then $0 = V\eta = \lim_{n \rightarrow \infty} VUf_n = \lim_{n \rightarrow \infty} f_n$. Now, if we put $\eta' := I^{-1}(C^{\frac{1}{2}}\eta)$ we obtain $0 = \lim_{n \rightarrow \infty} \|C^{\frac{1}{2}}Uf_n - C^{\frac{1}{2}}\eta\|_{\mathcal{H}} = \lim_{n \rightarrow \infty} \|U'f_n - \eta'\|_{\mathcal{H}'}$, from which $V'\eta' = \lim_{n \rightarrow \infty} V'U'f_n = \lim_{n \rightarrow \infty} f_n = 0$ follows. \square

The next result ensures the isomorphy of \mathcal{H}'_s and \mathcal{H}'_c , if and only if $t' = t_s$.

Proposition 2.9. *Let t' be a singular form satisfying $t_s \leq t' \leq t$. It follows $P_s C^{\frac{1}{2}} = C^{\frac{1}{2}} P_s = P_c$, where C is defined in (2.3), and hence $C^{\frac{1}{2}}(\mathcal{H}'_s) = \mathcal{H}'_c \subseteq I(\mathcal{H}'_s)$. Moreover, $\mathcal{H}'_s = I(\mathcal{H}'_s)$ if and only if $t_s = t'$, in which case one has $P_s Uf = I P'_s f \forall f \in E$.*

Proof: The proof is given in several steps:

(I) Let t' be any positive form with $t' \leq t$. Since $\|f\| \leq \|U'f\|_{\mathcal{H}'} \leq \|Uf\|_{\mathcal{H}} \forall f \in E$, there is a selfadjoint operator A on \mathcal{H} with

$$0 \leq A \leq C \leq \mathbf{1}_{\mathcal{H}} \quad (2.4)$$

and $\langle f, g \rangle = (Uf|AUg)_{\mathcal{H}} \forall f, g \in E$. Let $J: \mathcal{H} \rightarrow \mathcal{H}$ be the isometry defined by $Jf := A^{\frac{1}{2}}Uf \forall f \in E$. Since $A^{\frac{1}{2}}Uf = Jf = JVUf$ it follows $A^{\frac{1}{2}} = JV$ and therefore $\ker(A^{\frac{1}{2}}) = \ker(V) = \mathcal{H}'_s$. Consequently $A^{\frac{1}{2}}P_c = P_c A^{\frac{1}{2}} = A^{\frac{1}{2}}$. From (2.4) it follows $\|A^{\frac{1}{2}}\xi\|_{\mathcal{H}} \leq \|C^{\frac{1}{2}}\xi\|_{\mathcal{H}} \forall \xi \in \mathcal{H}$, and we obtain

$$\ker(C^{\frac{1}{2}}) \subseteq \ker(A^{\frac{1}{2}}) = \mathcal{H}'_s. \quad (2.5)$$

(II) Now let $t_s \leq t' \leq t$. It follows $P_s \leq C$. From $0 \leq P_s \leq C \leq \mathbf{1}_{\mathcal{H}}$ we get by [19], Proposition I.6.3 $0 \leq P_s \leq C^{\frac{1}{2}} \leq \mathbf{1}_{\mathcal{H}}$ and by [13], Proposition 2.2.13(c) $P_s = P_s C^{\frac{1}{2}} P_s$. Hence

$$C^{\frac{1}{2}}P_s = P_s C^{\frac{1}{2}} = P_s. \quad (2.6)$$

From (2.5) it follows $\ker(C^{\frac{1}{2}}) = \{0\}$ and $\overline{C^{\frac{1}{2}}(\mathcal{H})} = \ker(C^{\frac{1}{2}})^\perp = \mathcal{H}$. Consequently I is unitary. Let $\tilde{P}'_s := I P'_s I^*$ and $\tilde{P}'_c := I P'_c I^*$. Then $\tilde{P}'_s + \tilde{P}'_c = \mathbf{1}_{\mathcal{H}}$.

(III) Now, if t' is singular and $t_s \leq t' \leq t$ we have, using (2.2) for the form t' :

$$\begin{aligned} (Uf|AUg)_{\mathcal{H}} &= \langle f, g \rangle = (U'f|P'_c U'g)_{\mathcal{H}'} \\ &= (IU'f|IP'_c I^* IU'g)_{\mathcal{H}} \\ &= (C^{\frac{1}{2}}Uf|\tilde{P}'_c C^{\frac{1}{2}}Ug)_{\mathcal{H}}, \end{aligned}$$

from which, with $C = C^{\frac{1}{2}}(\tilde{P}'_c + \tilde{P}'_s)C^{\frac{1}{2}}$, we obtain the relations

$$A = C^{\frac{1}{2}}\tilde{P}'_c C^{\frac{1}{2}} \quad \text{and} \quad C - A = C^{\frac{1}{2}}\tilde{P}'_s C^{\frac{1}{2}}. \quad (2.7)$$

Using (2.5) and (2.6) we get $P_s = P_s(C - A)P_s = P_s C^{\frac{1}{2}}\tilde{P}'_s C^{\frac{1}{2}}P_s = P_s \tilde{P}'_s P_s$, from which we get

$$P_s = P_s \tilde{P}'_s = \tilde{P}'_s P_s. \quad (2.8)$$

(IV) Let $t_s = t'$. Since $\langle f, g \rangle + t(f, g) = (\langle f, g \rangle + t_c(f, g)) + (\langle f, g \rangle + t_s(f, g)) - \langle f, g \rangle$ we obtain from (2.2) and (2.3) $(Uf|Ug)_{\mathcal{H}} = (Uf|P'_c U'g)_{\mathcal{H}'} + (Uf|CUg)_{\mathcal{H}} - (Uf|AUg)_{\mathcal{H}}$, and therefore $\mathbf{1}_{\mathcal{H}} = P'_c + C - A$. Hence $P_s = C - A$. Thus, multiplying (2.7) with P_c if follows $0 = C^{\frac{1}{2}}P_c \tilde{P}'_s C^{\frac{1}{2}}$. Since $\ker(C^{\frac{1}{2}}) = \{0\}$ and $\overline{C^{\frac{1}{2}}(\mathcal{H})} = \mathcal{H}$ we

derive $0 = P_c \tilde{P}'_s$. Consequently by (2.8)

$$P_s = \tilde{P}'_s. \quad (2.9)$$

Now with (2.6) we have $P_s = P_s C^{\frac{1}{2}} = \tilde{P}'_s C^{\frac{1}{2}}$ and one deduces $P_s Uf = \tilde{P}'_s C^{\frac{1}{2}} Uf = IP'_s Uf \forall f \in E$.

(V) Conversely, let $\mathcal{K}'_s = I(\mathcal{K}'_s)$, then $P_s I = IP'_s$, that is (2.9). Thus, by (2.7) and (2.6) $C - A = P_s$ and the assertion follows. \square

Proof of Lemma 2.7: Since t' is closable we have by Lemma 2.2: $\mathcal{K}'_s = \{0\}$. Hence by Lemma 2.8: $\mathcal{K}'_s \subseteq \ker(C^{\frac{1}{2}})$. From (2.5) one gets $\ker(C^{\frac{1}{2}}) = \mathcal{K}'_s$. Consequently $P_c C^{\frac{1}{2}} = C^{\frac{1}{2}} P_c = C^{\frac{1}{2}}$, from which $\|Uf\|_{\mathcal{K}'} = \|C^{\frac{1}{2}} Uf\|_{\mathcal{K}} = \|C^{\frac{1}{2}} P_c Uf\|_{\mathcal{K}} \leq \|P_c Uf\|_{\mathcal{K}}$ follows. \square

Also there is a useful result for determining the decomposition $t = t_c + t_s$:

Lemma 2.10. *Let $t = t' + t''$, where t' and t'' are arbitrary positive forms such that $t_s \leq t''$ (or equivalently $t' \leq t_c$). Then the following conditions are equivalent:*

- (i) $t'' = t_s$ (and therefore $t' = t_c$);
- (ii) t'' is singular with respect to the scalar product $\langle \cdot | \cdot \rangle_{t'}$, where $\langle f | g \rangle_{t'} := \langle f, g \rangle + t'(f, g)$ for all $f, g \in E$.

Remark: If t' is closable, the condition $t_s \leq t''$ is fulfilled by Lemma 2.7.

Proof: (i) \Rightarrow (ii): Let $f \in E$ and $(f_n)_{n \in \mathbb{N}}$ a sequence in E with $\lim_{n \rightarrow \infty} \|Uf_n - P_c Uf\|_{\mathcal{K}} = 0$. Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} (\|f_n - f\|^2 + t_c(f_n - f, f_n - f)) \\ = \lim_{n \rightarrow \infty} \|P_c Uf_n - P_c Uf\|_{\mathcal{K}}^2 = 0 \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} t_s(f_n, f_n) = \lim_{n \rightarrow \infty} \|P_s Uf_n\|_{\mathcal{K}}^2 = 0.$$

Hence (ii) follows from Definition 2.1.

(ii) \Rightarrow (i): By Definition 2.1, for each $f \in E$ there is a sequence $(f_n)_{n \in \mathbb{N}}$ in E such that

$$\lim_{n \rightarrow \infty} (\|f_n - f\|^2 + t'(f_n - f, f_n - f)) = 0$$

and $\lim_{n \rightarrow \infty} t''(f_n, f_n) = 0$. Consequently $(Uf_n)_{n \in \mathbb{N}}$ is Cauchy in \mathcal{K} , converging to an $\eta \in \mathcal{K}$. Since $t_s \leq t''$, we have

$$0 = \lim_{n \rightarrow \infty} t_s(f_n, f_n) = \lim_{n \rightarrow \infty} \|P_s Uf_n\|_{\mathcal{K}}^2 = \|P_s \eta\|_{\mathcal{K}}^2.$$

Because of $\lim_{n \rightarrow \infty} t'(f_n, f_n) = t'(f, f)$ we get

$$\begin{aligned} \|f\|^2 + t_c(f, f) &= \lim_{n \rightarrow \infty} \|P_c Uf_n\|_{\mathcal{K}}^2 = \|\eta\|_{\mathcal{K}}^2 \\ &= \lim_{n \rightarrow \infty} \|Uf_n\|_{\mathcal{K}}^2 = \lim_{n \rightarrow \infty} (\|f_n\|^2 + t(f_n, f_n)) \\ &= \lim_{n \rightarrow \infty} (\|f_n\|^2 + t'(f_n, f_n) + t''(f_n, f_n)) \\ &= \|f\|^2 + t'(f, f). \end{aligned}$$

Hence $t_c(f, f) = t'(f, f)$. \square

We now turn to some examples which are based on two measures mutually singular to each other. By a unique lifting from an L^2 -space to a function space E we mean that in each L^2 -equivalence class there is a most one element of E .

Proposition 2.11. *Let X be a set, \mathfrak{B} a σ -algebra on X and μ and ν two measures on \mathfrak{B} singular to each other. Let E be a vector space of \mathbb{K} -valued functions on X so that $L^2(X, \mu)$ lifts uniquely to E , and E is dense in $L^2(X, \mu + \nu)$. Then E is dense in $L^2(X, \mu)$ and the positive form $t: E \times E \rightarrow \mathbb{K}, (f, g) \mapsto \int_X \bar{f}g \, d\nu$ is singular in $L^2(X, \mu)$ (that is, singular with respect to the scalar product of $L^2(X, \mu)$).*

Proof: That E is dense in $L^2(\mu)$ follows from $\|h\|_{\mu+\nu}^2 = \|h\|_{\mu}^2 + \|h\|_{\nu}^2$. Since ν is singular to μ , by definition there is a set $Y \in \mathfrak{B}$ such that $\mu(X \setminus Y) = 0$ and $\nu(Y) = 0$. Now let $g \in L^2(\mu)$ and choose a representant \tilde{g} of the equivalence class g , such that $\tilde{g}(x) = 0 \forall x \in X \setminus Y$. We regard \tilde{g} as an element of $L^2(\mu + \nu)$. Since E is dense in $L^2(\mu + \nu)$ there is a sequence $(f_n)_{n \in \mathbb{N}}$ in E with $\lim_{n \rightarrow \infty} \|f_n - \tilde{g}\|_{\mu+\nu} = 0$. Consequently $\lim_{n \rightarrow \infty} \|f_n - \tilde{g}\|_{\mu} = 0$ and $\lim_{n \rightarrow \infty} \|f_n - \tilde{g}\|_{\nu} = 0$. But since $\tilde{g} = 0$ ν -almost everywhere, we have $\lim_{n \rightarrow \infty} t(f_n, f_n) = \lim_{n \rightarrow \infty} \|f_n - \tilde{g}\|_{\nu}^2 = 0$. The assertion now follows from Definition 2.1. \square

In the situation of Proposition 2.11 one has $\langle f, g \rangle_{\mu} + t(f, g) = \int_X \bar{f}g \, d(\mu + \nu)$ and thus U is the canonical embedding of E into $L^2(X, \mu + \nu) = \mathcal{K}$. Hence $V: L^2(X, \mu + \nu) \rightarrow L^2(X, \mu)$ is the identity map with the kernel $\ker(V) = \mathcal{K}'_s = \{f \in \mathcal{K} \mid f(x) = 0 \forall x \in X \setminus Y\}$ which is isomorphic to $L^2(X, \nu)$.

Decompositions into the closable and singular parts on L^2 -spaces one can obtain by multiplication operators and singular measures. We state an example relevant for the Bose-Einstein condensation.

Example 2.12. Let E be the Fourier transform of the set of all infinitely differentiable \mathbb{C} -valued functions on \mathbb{R}^m with compact support and ν a finite Borel measure singular to the Lebesgue measure λ on \mathbb{R}^m . Further let $\varphi \in L^2(\mathbb{R}^m, \lambda)$ and M_φ the corresponding multiplication operator in $L^2(\mathbb{R}^m, \lambda)$. Then the form $t_c: E \times E \rightarrow \mathbb{C}$, $(f, g) \mapsto \langle M_\varphi f, M_\varphi g \rangle_\lambda$ is the closable part and $t_s: E \times E \rightarrow \mathbb{C}$, $(f, g) \mapsto \int_{\mathbb{R}^m} \bar{f}g \, d\nu$ is the singular part of the positive form $t := t_c + t_s$ in $L^2(\mathbb{R}^m, \lambda)$. Moreover, $\mathcal{K} = L^2(\mathbb{R}^m, \lambda + |\varphi|^2 \lambda + \nu)$ with $\mathcal{K}_c = L^2(\mathbb{R}^m, \lambda + |\varphi|^2 \lambda)$ and $\mathcal{K}_s = L^2(\mathbb{R}^m, \nu)$.

Proof: The measure $|\varphi|^2 \lambda + \nu$ is finite. Hence by standard arguments E is dense in $L^2(\lambda + |\varphi|^2 \lambda + \nu)$. Since ν is singular to $\lambda + |\varphi|^2 \lambda$, by Proposition 2.11 the form t_s is singular with respect to the scalar product $\langle f | g \rangle := \langle f, g \rangle_\lambda + t_c(f, g)$. The assertion follows from Lemma 2.10 and

$$\langle f, g \rangle_\lambda + t(f, g) = \int_{\mathbb{R}^m} \bar{f}g \, d(\lambda + |\varphi|^2 \lambda + \nu). \quad \square$$

3. Gauge-invariant Quasi-free States on the Weyl Algebra

3.1. Preliminaries

Let $\mathcal{W}(E)$ be the Weyl algebra over the complex pre-Hilbert space E , the unique C^* -algebra generated by nonzero elements $W(f)$, $f \in E$, satisfying the Weyl relations

$$W(f)W(g) = e^{-\frac{i}{2}\Im\langle f, g \rangle} W(f+g) \quad (3.1)$$

and

$$W(f)^* = W(-f)$$

for all $f, g \in E$ (see e.g. [12] or [13], Theorem 5.2.8). We mention, that $\mathcal{W}(E)$ is simple and thus has only faithful nontrivial representations. The weak*-compact convex set of all states on $\mathcal{W}(E)$ is denoted by \mathcal{S} . In the GNS-representation $(\Pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ of the state ω on $\mathcal{W}(E)$ we set $\mathcal{M}_\omega := \Pi_\omega(\mathcal{W}(E))''$ for the generated von Neumann algebra, where $''$ denotes the bicommutant.

A state ω on $\mathcal{W}(E)$ is called regular, if the unitary groups $(\Pi_\omega(W(tf)))_{t \in \mathbb{R}}$ are strongly continuous for all $f \in E$. An equivalent condition for a state ω to be regular is the continuity of $t \in \mathbb{R} \mapsto \omega(W(tf))$ for each $f \in E$. The infinitesimal generators – the field operators – of these unitary groups will be denoted by $\Phi_\omega(f)$, that is $\Pi_\omega(W(tf)) = e^{it\Phi_\omega(f)} \, \forall t \in \mathbb{R}$. We have for each real linear subspace $M \subseteq E$ and each $n \in \mathbb{N}$

$$\bigcap_{f \in M} \mathcal{D}(\Phi_\omega(f)^n) = \bigcap_{f_1, \dots, f_n \in M} \mathcal{D}(\Phi_\omega(f_1) \dots \Phi_\omega(f_n)),$$

which is dense in \mathcal{H}_ω , if M is finite dimensional. For a regular state ω one can introduce annihilation and creation operators

$$a_\omega(f) := \frac{1}{\sqrt{2}} (\Phi_\omega(f) + i\Phi_\omega(if)),$$

$$a_\omega^*(f) := \frac{1}{\sqrt{2}} (\Phi_\omega(f) - i\Phi_\omega(if)).$$

They are densely defined with domain $\mathcal{D}(\Phi_\omega(f)) \cap \mathcal{D}(\Phi_\omega(if))$, they are closed, it is $a_\omega(f)^* = a_\omega^*(f)$, $f \mapsto a_\omega(f)$ is antilinear and $f \mapsto a_\omega^*(f)$ is linear and they fulfill the canonical commutation relations (CCR) on a dense domain ([13], Lemma 5.2.12):

$$[a_\omega(f), a_\omega(g)] = [a_\omega^*(f), a_\omega^*(g)] = 0,$$

$$[a_\omega(f), a_\omega^*(g)] = \langle f, g \rangle \mathbf{1} \quad \forall f, g \in E.$$

$\omega \in \mathcal{S}$ is called analytic, whenever for each $f \in E$ the function $t \in \mathbb{R} \mapsto \omega(W(tf))$ is analytic in an open neighborhood of the origin. Usually for analytic states ω one defines

$$\omega(\Phi_\omega(f_1) \dots \Phi_\omega(f_n)) := \langle \Omega_\omega, \Phi_\omega(f_1) \dots \Phi_\omega(f_n) \Omega_\omega \rangle.$$

Now let \mathcal{J} be an arbitrary set and F a function from the nonempty ordered finite subsets of \mathcal{J} to the complex numbers. By the recursion relations $F(I) := \sum_{P_I} \prod_{J \in P_I} F_T(J)$ one constructs the truncated function F_T , where the sum is over all partitions P_I of the finite set $I \subseteq \mathcal{J}$ into ordered subsets. For an analytic state ω on $\mathcal{W}(E)$ this procedure defines a truncation ω_T , which satisfies

$$\omega_T(\Phi_\omega(f)) = \omega(\Phi_\omega(f))$$

$$\omega_T(\Phi_\omega(f_1); \Phi_\omega(f_2)) = \omega(\Phi_\omega(f_1) \Phi_\omega(f_2))$$

$$- \omega(\Phi_\omega(f_1)) \omega(\Phi_\omega(f_2)) \quad \text{etc.}$$

Hence, since ω is linear, the truncations $\omega_T(\cdot; \dots; \cdot)$ can be extended to multilinear functionals on the linear combinations of the field operators (see e.g. [13], p. 40). By the usual rules of differentiation and Taylor's theorem for holomorphic functions one easily deduces that for each $f \in E$ there is a $\delta_f > 0$ such that

$$\omega(W(tf)) = \sum_{n=0}^{\infty} \frac{i^n t^n}{n!} \omega(\Phi_\omega(f)^n)$$

$$= \exp \left\{ \sum_{n=1}^{\infty} \frac{i^n t^n}{n!} \omega_T(\Phi_\omega(f)^n) \right\}, \quad (3.2)$$

where all the series converge for $t \in]-\delta_f, \delta_f[$.

An analytic state ω is called quasi-free if $\omega_T(\Phi_\omega(f_1); \dots; \Phi_\omega(f_n)) = 0$ for all $n > 2$ and all $f_1, \dots, f_n \in E$. The notion of a state to be quasi-free was first defined by Robinson [14] and an equivalent definition was given in [15].

Lemma 3.1. *Let $\omega \in \mathcal{S}$. The following conditions are equivalent:*

- (i) ω is quasi-free;
- (ii) ω is analytic and $\omega_T(\Phi_\omega(f)^n) = 0$ for all $n > 2$ and all $f \in E$;
- (iii) for each $f \in E$ there is a polynomial $P_f: \mathbb{R} \rightarrow \mathbb{C}$ such that $\omega(W(tf)) = \exp\{P_f(t)\}$ $\forall t \in \mathbb{R}$.

If one of these conditions is fulfilled, ω is entire analytic and $P_f(t) = it \omega(\Phi_\omega(f)) - \frac{t^2}{2} [\omega(\Phi_\omega(f)^2) - \omega(\Phi_\omega(f))^2]$ for all $t \in \mathbb{R}$ and all $f \in E$.

Proof: (i) \Rightarrow (ii) is trivial. (ii) \Rightarrow (i) and (ii) \Rightarrow (iii): Using (3.2) we get

$$\omega(W(tf)) = \exp \left\{ it \omega_T(\Phi_\omega(f)) - \frac{t^2}{2} \omega_T(\Phi_\omega(f)^2) \right\}.$$

Now by (3.1) and by differentiating one gets

$$\begin{aligned} & \omega(\Phi_\omega(f_1) \dots \Phi_\omega(f_m)) \\ &= (-i)^m \frac{\partial}{\partial t_1} \dots \frac{\partial}{\partial t_m} \omega(W(t_1 f_1) \dots W(t_m f_m)) \Big|_{t_1 = \dots = t_m = 0} \\ &= \omega_T(\Phi_\omega(f_m)) \omega(\Phi_\omega(f_1) \dots \Phi_\omega(f_{m-1})) \\ &+ \sum_{k=1}^{m-1} \omega_T(\Phi_\omega(f_k); \Phi_\omega(f_m)) \omega(\Phi_\omega(f_1) \dots \\ &\dots \Phi_\omega(f_{k-1}) \Phi_\omega(f_{k+1}) \dots \Phi_\omega(f_{m-1})), \end{aligned}$$

from which (i) follows. The proof of (iii) \Rightarrow (i) is due to [16], we briefly refer: Since ω is a state, the function $\mathbb{R} \ni t \mapsto \omega(W(tf))$ is positive-definite. However, such that $\exp \circ P_f$ is positive-definite, the only possibility for the polynomial P_f is to have a degree less or equal two. Now compare with (3.2). \square

A state ω on $\mathcal{W}(E)$ is called gauge-invariant, if $\omega(W(e^{i\theta} f)) = \omega(W(f))$ for all $\theta \in \mathbb{R}$ and all $f \in E$. If ω is analytic, using (3.2) and the CCR, the gauge-invariance of ω is equivalent to $\omega(a_\omega^*(f)^m a_\omega(f)^n) = 0$ for all $m \neq n$ and all $f \in E$.

Using the CCR one gets

$$\Phi_\omega(f)^2 = \frac{1}{2} (a_\omega^*(f)^2 + a_\omega(f)^2 + 2a_\omega^*(f) a_\omega(f) + \|f\|^2 \mathbf{1}).$$

Hence by the Lemma 3.1 each gauge-invariant quasi-free state $\omega \in \mathcal{S}$ has the form

$$\omega(W(f)) = \exp \left\{ -\frac{1}{4} \|f\|^2 - \frac{1}{4} t(f, f) \right\} \quad \forall f \in E, \quad (3.3)$$

where $t: E \times E \rightarrow \mathbb{C}$ is the positive sesquilinear form defined by

$$t(f, g) = 2\omega(a_\omega^*(g) a_\omega(f)) = 2\omega_T(a_\omega^*(g); a_\omega(f)) \quad (3.4)$$

(t is positive since $t(f, f) = \|a_\omega(f)\|_{\Omega_\omega}^2 \geq 0$). Conversely, it is well known in the literature [15], [8] that for each positive form t on E there exists a unique gauge-invariant quasi-free $\omega \in \mathcal{S}$ satisfying (3.3) (compare also the remark in Subsection 3.2).

If $t \equiv 0$, then (3.3) determines the Fock state $\omega_{\mathcal{F}} \in \mathcal{S}$. Its GNS-representation is given by the Bose-Fock space $\mathcal{F}_+(\bar{E})$ over \bar{E} , the vacuum vector $\Omega_{\mathcal{F}} = (1, 0, 0, \dots)$ and the usual Fock representation $\Pi_{\mathcal{F}}(W(f)) = W_{\mathcal{F}}(f)$ (\bar{E} is the completion of $(E, \langle \cdot, \cdot \rangle)$).

Following the construction introduced in [4] and generalized in [15], the GNS-representation of the gauge-invariant quasi-free state ω corresponding to the closable positive form t is given by $\mathcal{H}_\omega = \mathcal{F}_+(\mathcal{V}_1) \otimes \mathcal{F}_+(\mathcal{V}_2)$, where $\mathcal{V}_1 = (\mathbf{1} + T)^{\frac{1}{2}}(E)$ and $\mathcal{V}_2 = JT^{\frac{1}{2}}(E)$ with a positive selfadjoint operator T on \bar{E} so that $\mathcal{D}(T^{\frac{1}{2}}) \supseteq E$ and $\frac{1}{2} t(f, g) = \langle T^{\frac{1}{2}} f, T^{\frac{1}{2}} g \rangle$ $\forall f, g \in E$ and an arbitrary antilinear involution J on \bar{E} satisfying $\langle Jf, Jg \rangle = \langle g, f \rangle$ $\forall f, g \in \bar{E}$, the cyclic vector $\Omega_\omega = \Omega_{\mathcal{F}} \otimes \Omega_{\mathcal{F}}$ and the representation $\Pi_\omega(W(f)) = W_{\mathcal{F}}((\mathbf{1} + T)^{\frac{1}{2}} f) \otimes W_{\mathcal{F}}(JT^{\frac{1}{2}} f)$. This representation is a factor, it is irreducible if and only if $T = 0$. It is normal to the Fock representation if and only if $T^{\frac{1}{2}}$ is Hilbert-Schmidt.

If \bar{t} is the closure of t we denote by $\bar{\omega}$ the canonical extension of ω to $\mathcal{W}(\mathcal{D}(\bar{t}))$:

$$\bar{\omega}(W(f)) := \exp \left\{ -\frac{1}{4} \|f\|^2 - \frac{1}{4} \bar{t}(f, f) \right\} \quad \forall f \in \mathcal{D}(\bar{t}). \quad (3.5)$$

Lemma 3.2. *It is $(\Pi_\omega, \mathcal{H}_\omega, \Omega_\omega) = (\Pi_{\bar{\omega}}|_{\mathcal{W}(E)}, \mathcal{H}_{\bar{\omega}}, \Omega_{\bar{\omega}})$ and $\mathcal{M}_\omega = \mathcal{M}_{\bar{\omega}}$.*

Proof: Using the Weyl relations (3.1) one gets for $f, g, h \in \mathcal{D}(\bar{t})$

$$\begin{aligned} & \|(\Pi_{\bar{\omega}}(W(f)) - \Pi_{\bar{\omega}}(W(g))) \Pi_{\bar{\omega}}(W(h)) \Omega_{\bar{\omega}}\|^2 \\ &= \bar{\omega}(W(-h) [W(-f) - W(-g)] [W(f) - W(g)] W(h)) \\ &= 2 - 2 \Re(e^{i(\frac{1}{2} \Im \langle f, g \rangle + \Im \langle h, g-f \rangle)}) \\ &\quad \cdot \exp \left\{ -\frac{1}{4} (\|f-g\|^2 - \bar{t}(f-g, f-g)) \right\}. \end{aligned}$$

Thus, if $\mathcal{D}(\bar{t})$ is equipped with the norm $\|f\|_{\bar{t}}^2 := \|f\|^2 + t(f, f)$ (compare (2.1)), the map $\mathcal{D}(\bar{t}) \ni f \mapsto \Pi_{\bar{\omega}}(W(f))$ is continuous in the strong operator topology. \square

3.2. The Central Decomposition of Gauge-invariant Quasi-free States

In this subsection let ω be a fixed gauge-invariant quasi-free state on the Weyl algebra $\mathcal{W}(E)$ determined by (3.3) with the positive form $t: E \times E \rightarrow \mathbb{C}$. As in Section 2 let $U: E \rightarrow \mathcal{H}$ be the injection of E into its completion with the scalar product (2.1) and $V: \mathcal{H} \rightarrow \bar{E}$ the continuous extension of U^{-1} and $t = t_c + t_s$ the decomposition of t into its closable and singular part with closable subspace \mathcal{H}_c and singular subspace \mathcal{H}_s . Denote by ω_c the gauge-invariant quasi-free state defined by (3.3) with the form t_c and $\bar{\omega}_c$ its canonical extension (3.5).

If we regard E as an additive group equipped with the discrete topology, the associated character group \hat{E} becomes a compact abelian group with respect to the so called Δ -topology (cf. [17], (23.13)). For each $\chi \in \hat{E}$ there is a $*$ -automorphism τ_χ on $\mathcal{W}(E)$ so that $\tau_\chi(W(f)) = \chi(f) W(f) \forall f \in E$. If $\hat{\mathcal{H}}_s$ is the character group of the discrete group $(\mathcal{H}_s, +)$, each $\chi \in \hat{\mathcal{H}}_s$ lifts to a $\chi' := \chi \circ P_s \circ U \in \hat{E}$. Define

$$N_\omega := \{\omega_c \circ \tau_{\chi'} \mid \chi \in \hat{\mathcal{H}}_s\} \subseteq \mathcal{S}$$

and

$$q_\omega: \hat{\mathcal{H}}_s \rightarrow \mathcal{S}, \quad \chi \mapsto \omega_c \circ \tau_{\chi'}.$$

Because $\lim_{i \in I} \chi_i = \chi$ in the Δ -topology of $\hat{\mathcal{H}}_s$ is equivalent to $\lim_{i \in I} \chi_i(\xi) = \chi(\xi) \forall \xi \in \mathcal{H}_s$ ([17], (23.15)), it follows

$$\begin{aligned} \lim_{i \in I} q_\omega(\chi_i)(W(f)) &= \lim_{i \in I} \chi_i(P_s U f) \omega_c(W(f)) \\ &= \chi(P_s U f) \omega_c(W(f)) = q_\omega(\chi)(W(f)), \end{aligned}$$

which gives $\lim_{i \in I} q_\omega(\chi_i) = q_\omega(\chi)$ in the weak*-topology, showing q_ω to be continuous and N_ω to be compact since $\hat{\mathcal{H}}_s$ is so.

Because $\mathcal{H}_s \ni \xi \mapsto \exp\{-\frac{1}{4} \|\xi\|_{\mathcal{H}}^2\}$ is positive-definite, by Bochner's theorem ([17], (33.3)) there is a unique measure $\mu \in M^+(\hat{\mathcal{H}}_s)$ (the finite positive regular Borel measures on $\hat{\mathcal{H}}_s$), such that

$$\exp\{-\frac{1}{4} \|\xi\|_{\mathcal{H}}^2\} = \int_{\hat{\mathcal{H}}_s} \chi(\xi) d\mu(\chi) \quad \forall \xi \in \mathcal{H}_s. \quad (3.6)$$

Let us transfer μ to a Borel measure μ_ω on \mathcal{S} by setting

$$\mu_\omega(B) := \mu(q_\omega^{-1}(B)) \quad \text{for each Borel set } B \subseteq \mathcal{S}. \quad (3.7)$$

As a consequence $\mu_\omega(\mathcal{S} \setminus N_\omega) = 0$ and the spaces $L^p(\hat{\mathcal{H}}_s, \mu)$ and $L^p(\mathcal{S}, \mu_\omega)$ can be identified for each $p \in [1, \infty]$.

Lemma 3.3. *The measure μ_ω is regular.*

Proof: Let B be a Borel subset of \mathcal{S} . Since μ is regular and $q_\omega(K)$ is compact, if K is compact, we get

$$\begin{aligned} \mu_\omega(B) &= \mu(q_\omega^{-1}(B)) \\ &= \sup\{\mu(K) \mid K \subseteq q_\omega^{-1}(B), K \text{ compact}\} \\ &= \sup\{\mu_\omega(q_\omega(K)) \mid q_\omega(K) \subseteq B, K \text{ compact}\} \\ &\leq \sup\{\mu_\omega(K) \mid K \subseteq B, K \text{ compact}\} \\ &\leq \mu_\omega(B). \end{aligned}$$

That is: μ_ω is inner regular. Now let $A^c := \mathcal{S} \setminus A$. Since \mathcal{S} is compact, each closed subset $K \subseteq \mathcal{S}$ is compact. From the inner regularity of μ_ω we get

$$\begin{aligned} \mu_\omega(\mathcal{S}) - \mu_\omega(B) &= \mu_\omega(B^c) \\ &= \sup\{\mu_\omega(K) \mid K \subseteq B^c, K \text{ closed}\} \end{aligned}$$

and therefore

$$\begin{aligned} \mu_\omega(B) &= \mu_\omega(\mathcal{S}) - \mu_\omega(B^c) \\ &= \inf\{\mu_\omega(\mathcal{S}) - \mu_\omega(K) \mid K \subseteq B^c, K \text{ closed}\} \\ &= \inf\{\mu_\omega(K^c) \mid K^c \supseteq B, K \text{ closed}\} \\ &= \inf\{\mu_\omega(U) \mid B \subseteq U, U \text{ open}\}. \end{aligned} \quad \square$$

Since

$$\begin{aligned} \omega(W(f)) &= \omega_c(W(f)) \exp(-\frac{1}{4} \|P_s U f\|_{\mathcal{H}}^2) \\ &= \int_{\hat{\mathcal{H}}_s} \omega_c \circ \tau_{\chi'}(W(f)) d\mu(\chi) \end{aligned}$$

we arrive at the integral decomposition of our gauge-invariant quasi-free state $\omega \in \mathcal{S}$

$$\omega = \int_{\hat{\mathcal{H}}_s} \omega_c \circ \tau_{\chi'} d\mu(\chi) = \int_{\mathcal{S}} \varphi d\mu_\omega(\varphi). \quad (3.8)$$

Remark: Using similar arguments one can prove that indeed each positive form $t: E \times E \rightarrow \mathbb{C}$ by (3.3) defines a gauge-invariant quasi-free state on $\mathcal{W}(E): E \ni f \mapsto \exp\{-\frac{1}{4} t(f, f)\}$ is positive-definite and determines a measure $\varrho \in M^+(\hat{E})$ which lifts to a regular measure ϱ_t on \mathcal{S} via the map $p_t(\chi) := \omega_{\mathcal{F}} \circ \tau_\chi$. Thus $\omega := \int_{\mathcal{S}} \varphi d\varrho_t(\varphi)$ defines a state on $\mathcal{W}(E)$ satisfying (3.3).

Identifying the elements of $L^\infty(\hat{\mathcal{H}}_s, \mu)$ with multiplication operators on $L^2(\hat{\mathcal{H}}_s, \mu)$ and introducing for each $\eta \in \mathcal{H}_s$ the function

$$F_\eta: \hat{\mathcal{H}}_s \rightarrow \mathbb{C}, \quad \chi \mapsto \chi(\eta)$$

we get a result, which is a slight generalization of one in [4] and analogous to one in [18], [8]. Observing Lemma 3.2 and Proposition 2.6 we have with $V = VP_c$:

Proposition 3.4. Let $W_\omega(\xi) := \Pi_{\overline{\omega}_c}(W(V\xi)) \otimes F_{R_\xi}$, $\xi \in \mathcal{K}$. Then the GNS-representation of ω is given by

$$\mathcal{H}_\omega = \mathcal{H}_{\omega_c} \otimes L^2(\hat{\mathcal{K}}_s, \mu), \quad \Omega_\omega = \Omega_{\omega_c} \otimes \iota,$$

$$\Pi_\omega(W(f)) = W_\omega(Uf) \quad \forall f \in E$$

with $\iota(\chi) \equiv 1$. Moreover, $\mathcal{M}_\omega = \mathcal{M}_{\omega_c} \overline{\otimes} L^\infty(\hat{\mathcal{K}}_s, \mu)$ and the map $W_\omega(\cdot): \mathcal{K} \rightarrow \mathcal{M}_\omega$ is continuous with respect to the norm of \mathcal{K} and the strong operator topology of \mathcal{M}_ω .

Proof: The continuity of $W_\omega(\cdot)$ follows from the proof of Lemma 3.2 and from a similar argument for the state $\langle \iota, \cdot \rangle$ on $\mathcal{C}(\hat{\mathcal{K}}_s)$ (the continuous functions on $\hat{\mathcal{K}}_s$), regarded as multiplication operators, and observing that $\text{LH}\{F_\eta | \eta \in \mathcal{K}_s\}$ is dense in $\mathcal{C}(\hat{\mathcal{K}}_s)$ by the Stone-Weierstraß theorem. Now, if $\xi \in \mathcal{K}$, then $\lim_{n \rightarrow \infty} \|Uf_n - \xi\|_{\mathcal{K}} = 0$ for some $f_n \in E$ and $s\text{-}\lim_{n \rightarrow \infty} W_\omega(Uf_n) = W_\omega(\xi)$. Hence, $W_\omega(\eta) = \mathbf{1}_{\omega_c} \otimes F_\eta \in \mathcal{M}_\omega$ for $\eta \in \mathcal{K}_s$ and $W_\omega(\overline{P_c U}f) = \Pi_{\overline{\omega}_c}(W(f)) \otimes \mathbf{1} \in \mathcal{M}_\omega$ for $f \in \mathcal{D}(\overline{t_c})$. Use [19], Theorem III-1.2, and the rest is obvious. \square

Thus, since \mathcal{M}_{ω_c} is a factor, the center of \mathcal{M}_ω is given by

$$\mathcal{Z}_\omega = \mathcal{M}_\omega \cap \mathcal{M}'_\omega = \mathbf{1}_{\omega_c} \otimes L^\infty(\hat{\mathcal{K}}_s, \mu). \quad (3.9)$$

Identifying $L^\infty(\hat{\mathcal{K}}_s, \mu)$ and $L^\infty(\mathcal{S}, \mu_\omega)$ (via the map q_ω) we introduce the Tomita map $\kappa_\mu: L^\infty(\hat{\mathcal{K}}_s, \mu) \rightarrow \mathcal{M}'_\omega$ defined for all $A \in \mathcal{W}(E)$ and $F \in L^\infty(\hat{\mathcal{K}}_s, \mu)$ by (cf. [13], Lemma 4.1.21)

$$\begin{aligned} \langle \Omega_\omega, \kappa_\mu(F) \Pi_\omega(A) \Omega_\omega \rangle &:= \int_{\hat{\mathcal{K}}_s} F(\chi) \omega_c \circ \tau_{\chi'}(A) d\mu(\chi) \\ &= \int_{\mathcal{S}} F \circ q_\omega^{-1}(\varphi) \varphi(A) d\mu_\omega(\varphi). \end{aligned} \quad (3.10)$$

Theorem 3.5. We have $\kappa_\mu(F) = \mathbf{1}_{\omega_c} \otimes F \quad \forall F \in L^\infty(\hat{\mathcal{K}}_s, \mu)$, and the measure μ_ω from (3.7) is the central measure of the gauge-invariant quasi-free state $\omega \in \mathcal{S}$.

Proof: We have for $\eta \in \mathcal{K}_s$ and all $f \in E$

$$\begin{aligned} \langle \Omega_\omega, W_\omega(\eta) \Pi_\omega(W(f)) \Omega_\omega \rangle &= \langle \Omega_{\omega_c} \otimes \iota, \Pi_{\overline{\omega}_c}(W(VUf)) \otimes F_{P_s Uf + \eta} \Omega_{\omega_c} \otimes \iota \rangle \\ &= \omega_c(W(f)) \int_{\hat{\mathcal{K}}_s} \chi(P_s Uf + \eta) d\mu(\chi) \\ &= \int_{\hat{\mathcal{K}}_s} F_\eta(\chi) \omega_c \circ \tau_{\chi'}(W(f)) d\mu(\chi), \end{aligned}$$

from which with (3.10) one gets $\kappa_\mu(F_\eta) = W_\omega(\eta) = \mathbf{1}_{\omega_c} \otimes F_\eta$. $\kappa_\mu(F) = \mathbf{1}_{\omega_c} \otimes F$ follows from the linearity and the $\sigma(L^\infty, L^1)$ -continuity of κ_μ . Hence κ_μ is a *-isomorphism from $L^\infty(\hat{\mathcal{K}}_s, \mu)$ onto \mathcal{Z}_ω and μ_ω is central by [13], Proposition 4.1.22. \square

Similar to (3.5) it is possible to extend ω to a larger Weyl algebra. For doing this we introduce on \mathcal{K} the symplectic form $\sigma_{\mathcal{K}}(\xi, \eta) := \Im \langle V\xi, V\eta \rangle$, $\chi, \eta \in \mathcal{K}$, which extends $\Im \langle f, g \rangle$, $f, g \in E$ continuously to $\mathcal{K} \times \mathcal{K}$. By Proposition 2.6 $\mathcal{K}_s = \{\xi \in \mathcal{K} \mid \sigma_{\mathcal{K}}(\xi, \eta) = 0 \quad \forall \eta \in \mathcal{K}\}$. Then ω extends to a state $\bar{\omega}$ on the Weyl algebra $\overline{A(\mathcal{K}, \sigma_{\mathcal{K}})}$ (see the appendix), satisfying $\bar{\omega}(\delta_\xi) = \exp\{-\frac{1}{4} \|\xi\|_{\mathcal{K}}^2\}$ $\forall \xi \in \mathcal{K}$. Moreover, by Proposition 2.6 and Lemma A.1 we have $\overline{A(\mathcal{K}, \sigma_{\mathcal{K}})} = \overline{A(\mathcal{K}_c, \sigma_{\mathcal{K}})} \otimes \overline{A(\mathcal{K}_s, \sigma_{\mathcal{K}})} \cong \mathcal{W}(\mathcal{D}(\overline{t_c})) \otimes \mathcal{C}(\hat{\mathcal{K}}_s)$ and $\bar{\omega}$ becomes the product state $\bar{\omega} \cong \overline{\omega_c} \otimes \mu$, where μ is from (3.6). Its GNS-representation is given by $\Pi_{\bar{\omega}}(\delta_\xi) = W_\omega(\xi)$, $\xi \in \mathcal{K}$ and $\mathcal{H}_{\bar{\omega}} = \mathcal{H}_\omega$ and $\Omega_{\bar{\omega}} = \Omega_\omega$.

If $(E, \langle \cdot, \cdot \rangle)$ is separable, each state $\varphi = \omega_c \circ \tau_{\chi'}$ occurring in the support of the measure μ_ω has the GNS-representation $(\Pi_\varphi, \mathcal{H}_{\omega_c}, \Omega_{\omega_c})$ with separable \mathcal{H}_{ω_c} and $\Pi_\varphi(W(f)) = \chi(P_s Uf) \Pi_{\omega_c}(W(f))$. Moreover one easily checks the measurability of the von Neumann algebra field $N_\omega \ni \varphi \mapsto \mathcal{M}_\varphi$ ([20], Chapitre II § 3). Hence the direct integral exists and by Effros' theorem ([13], Theorem 4.4.9) we get the spatial decomposition

$$(\Pi_\omega, \mathcal{H}_\omega, \Omega_\omega) = \bigoplus_{N_\omega} (\Pi_\varphi, \mathcal{H}_\varphi, \Omega_\varphi) d\mu_\omega(\varphi),$$

$$\mathcal{M}_\omega = \bigoplus_{N_\omega} \mathcal{M}_\varphi d\mu_\omega(\varphi).$$

Let \mathcal{K}_s be finite dimensional, $\dim(\mathcal{K}_s) = n$. We construct the measure μ explicitly. Set $\mathcal{K}_s \cong \mathbb{R}^{2n}$ and thus $\hat{\mathcal{K}}_s \cong \mathbb{R}^{2n}$, the set of all characters. We decompose the Gauß function of (3.6) via Fourier transformation on

$$\mathbb{R}^{2n}: \exp\{-\frac{1}{4} \|\xi\|^2\} = \int_{\mathbb{R}^{2n}} e^{i \langle x | \xi \rangle} d\varrho(x) \quad \forall \xi \in \mathbb{R}^{2n},$$

where $d\varrho(x) := \frac{1}{\pi^n} e^{-\|x\|^2} d\lambda(x)$ with the Lebesgue

measure λ and $\langle x | y \rangle$ is the usual scalar product on \mathbb{R}^{2n} . Obviously, the injective (but not surjective) map $C: (\mathbb{R}^{2n}, \langle \cdot | \cdot \rangle) \rightarrow \widehat{\mathbb{R}^{2n}}$, $x \mapsto \exp\{i \langle x | \cdot \rangle\}$ is continuous. Let μ be the image measure of ϱ under the map C

(constructed analogously to (3.7)) and $\mathbb{R}^{2n} = \bigcup_{m=1}^{\infty} K_m$ with compact K_m . By the continuity of C each $C(K_m)$

is compact and hence $C(\mathbb{R}^{2n}) = \bigcup_{m=1}^{\infty} C(K_m)$ is Borel

and supports μ . As in Lemma 3.3 one proves regularity of μ and hence $\mu \in M^+(\widehat{\mathbb{R}^{2n}})$ is the unique measure satisfying (3.6). Similarly one checks $q_\omega(C(\mathbb{R}^{2n})) \subseteq N_\omega$ to be Borel and supporting μ_ω . Remembering $C(x)' = C(x) \circ P_s \circ U \in \hat{E}$, additional to (3.8) we have

$$\omega = \int_{\mathbb{R}^{2n}} \omega_c \circ \tau_{C(x)} d\varrho(x). \quad (3.11)$$

One may ask if there is a similar structure also for infinite dimensional \mathcal{H}_s . This is the case if $P_s U(E)$ is a nuclear space and $\|\cdot\|_{\mathcal{H}}$ a continuous hilbertian norm on $P_s U(E)$. By the Bochner-Minlos theorem one gets a Gauß measure ϱ on the dual $P_s U(E)^*$, whose canonical image measure in $\widehat{P_s U(E)}$ agrees with $\mu \in M^+(\widehat{P_s U(E)})$ defined similar to (3.6).

Now let us turn to the example of Bose-Einstein condensation: E is the space of all Lebesgue square integrable functions on \mathbb{R}^m with compact support. The limiting Gibbs state ω^β above the critical particle density and at inverse temperature $\beta > 0$ is gauge-invariant and quasi-free and given by (3.3) with the positive form $t: E \times E \rightarrow \mathbb{C}$, $(f, g) \mapsto \langle T^{\frac{1}{2}} f, T^{\frac{1}{2}} g \rangle + \gamma \hat{f}(0) \hat{g}(0)$ in $L^2(\mathbb{R}^m, \lambda)$ (λ , Lebesgue measure) with $T := 2e^{\beta\Delta}(\mathbf{1} - e^{\beta\Delta})^{-1}$ (Δ , Laplacian) and $\gamma > 0$ and \hat{f} is the Fourier transform of f [4], [2], [13], [3]. Using Fourier transformation we are in the situation of Example 2.12 with the Dirac measure $\nu = \gamma \delta_0$. Hence, $t_c(f, g) := \langle T^{\frac{1}{2}} f, T^{\frac{1}{2}} g \rangle$ is the closable part and $t_s(f, g) := \gamma \hat{f}(0) \hat{g}(0) = \gamma \int \hat{f} \hat{g} d\delta_0$ is the singular part of t and $\mathcal{H}_s \cong L^2(\mathbb{R}^m, \gamma \delta_0) \cong \mathbb{C}$ with $P_s f = \sqrt{\gamma} \hat{f}(0)$. Setting $\hat{f}(0) \cong (\Re \hat{f}(0), \Im \hat{f}(0))$, from (3.11) one obtains the central decomposition of ω^β

$$\omega^\beta(W(f)) = \omega_c^\beta(W(f)) \frac{1}{\pi} \int_{\mathbb{R}^2} e^{i\sqrt{\gamma}\langle x | \hat{f}(0) \rangle} e^{-\|x\|^2} d^2x \quad \forall f \in E.$$

Acknowledgements

The author is indebted to Prof. Dr. A. Rieckers, J. Hertle and A. Rapp for useful discussions and is grateful to the Deutsche Forschungsgemeinschaft for financial support.

Appendix:

Weyl Algebra with Degenerate Symplectic Form

Let H be a real vector space and σ a (possibly degenerate) symplectic form from $H \times H$ into \mathbb{R} . Let

$\Delta(H, \sigma)$ be the complex vector space generated by the functions δ_x ($x \in H$) from H to \mathbb{C} defined by

$$\delta_x(y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}.$$

$\Delta(H, \sigma)$ is a $*$ -algebra with unit δ_0 with respect to the product $\delta_x \cdot \delta_y = e^{-\frac{i}{2}\sigma(x, y)} \delta_{x+y}$ and the involution $\delta_x^* = \delta_{-x}$. In [21], by means of the completion with the minimal regular norm the C^* -Weyl algebra $\overline{\Delta(H, \sigma)}$ is obtained. If H is a pre-Hilbert space and $\sigma(x, y) = \Im \langle x, y \rangle$, then $\overline{\Delta(H, \sigma)} = \mathcal{W}(E)$, the Weyl algebra of Subsection 3.1. From (2.17) and (3.2) in [21] is seen that each function $C: H \rightarrow \mathbb{C}$, so that $C(0) = 1$ and that the kernel $H \times H \ni (x, y) \mapsto e^{-\frac{i}{2}\sigma(x, y)} C(y-x)$ is positive-definite [22], defines a state ω on $\overline{\Delta(H, \sigma)}$, satisfying $\omega(\delta_x) = C(x) \forall x \in H$.

Lemma A.1. *If $\sigma \equiv 0$ and \hat{H} is the compact character group (Δ -topology [17], (23.13)) of the additive discrete group $(H, +)$, then there is a unique $*$ -isomorphism γ from the abelian C^* -algebra $\overline{\Delta(H, 0)} =: \overline{\Delta(H)}$ onto the continuous functions $\mathcal{C}(\hat{H})$ of \hat{H} so that $\gamma(\delta_x) = F_x \forall x \in H$, where $F_x(\chi) := \chi(x) \forall \chi \in \hat{H}$.*

Proof: Each $\chi \in \hat{H}$ is a positive-definite function on H and by the above arguments defines a state φ_χ on $\overline{\Delta(H)}$ with $\varphi_\chi(\delta_x) = \chi(x) \forall x \in H$. Obviously $\varphi_\chi \in \Sigma$, the weak*-compact Hausdorff space of all homomorphisms from $\overline{\Delta(H)}$ onto \mathbb{C} (Σ is the spectrum of $\overline{\Delta(H)}$). Since $\lim_{i \in I} \chi_i = \chi$ in the Δ -topology is equivalent to $\lim_{i \in I} \chi_i(x) = \chi(x) \forall x \in H$ ([17], (23.15)) and the latter is equivalent to $\lim_{i \in I} \varphi_{\chi_i} = \varphi_\chi$ in the weak*-topology, the map $\chi \mapsto \varphi_\chi$ is a homeomorphism from \hat{H} onto Σ . Hence $\mathcal{C}(\hat{H}) \cong \mathcal{C}(\Sigma)$ with $F_x \cong \tilde{\delta}_x$, where $\tilde{\delta}_x(\varphi) = \varphi(\delta_x) \forall \varphi \in \Sigma$. Now use the Gelfand transform, which ensures the isomorphism of $\overline{\Delta(H)}$ and $\mathcal{C}(\Sigma)$ with $\delta_x \cong \tilde{\delta}_x$ ([19], Theorem I.4.4). \square

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